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SOME PROPERTIES OF THE BILEVEL PROGRAMMING PROBLEM, EXISTENCE AND APPLICATIONS

by

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1 Introduction

Many papers have been devoted to the bilevel programming problem in the last decade. All of them recognize that this type of problem is nonconvex and very difficult to solve. In particular, the linear problem has received more attention and has been solved at least partially, since some problems have been encountered when considering algorithms.

The value function technique plays a central role in sensitivity analysis, controllability and even in establishing necessary conditions, see for example [10], [11], [12], [13], [17] and [18]. In the present paper however, we will use the value function to express the second level of the $(BLPP)$, an abreviation to: *bi-level programming problem.* The advantage of using the value function is to transform the $(BLPP)$ to a one level programming problem containing this function in one of the inequality constraints. However, one disadvantage is the implicit hypothesis that the $(BLPP)$ has some kind of cooperation (see remark 3.1 (2)). Another disadvantage is the possibility that the value function can have bad behavior with respect to finiteness, continuity or differentiabilty. Moreover, the value function itself express a parameterized programming problem, which can be difficult to solve. In spite of these disadvantages, the formulation considered here will show its efficiency in many typical problems; we will consider only three examples. Other examples are examined in another paper on necessary conditions, [19].

The present paper is organized as follows. In the next section, we formulate the problem in a standard form. Section 3 is devoted to the reformulation of the $(BLPP)$ by means of the value function and to the study of some properties concerning this function and the set of solutions to the second level. We show in section 4 why the problem has a nonconvex nature and at the same time we identify a class of convex problems. Section 5 is reserved to the existence theory, where two examples are examined in details. In section 6 the multi-level programming problem is sddressed and an example is given. Finally, section 7 is devoted to the study of a concrete application of bi-level programming to a Lanchester model on market share competition.

2 Formulation of the problem

Although we are intersted to the study of $(BLPP)$ in this article, we can address the general $(MLPP)$ *multi-level programming problem* and generalize results of oncomming sections to this class of problem. A formulation of the $(MLPP)$ is given in section 6.

Let n_i, m_i $(i = 1, 2)$ be integers with $n_i \geq 1$ and $m_i \geq 0$. We are given functions: $F, f : R^{n_1} \times R^{n_2} \to R, G = [G_1, \ldots, G_{m_2}] : R^{n_1} \times R^{n_2} \to R^{m_2}, g = [g_1, \ldots, g_{m_1}] : R^{n_1} \times R^{n_2} \to R^{m_1}$ and sets $X \subset R^{n_1}, Y \subset R^{n_2}$. The standard bilevel programming problem is stated as follows:

$(BLPP)$

$$(P_1) \begin{cases} \min\limits_{x \in X} F(x, y), \\ \\ \text{s.t.: } G(x, y) \leq 0, \end{cases}$$

s.t. for each fixed x in $X, y = y(x)$ is a solution to the problem:

$$(P_2) \begin{cases} \min\limits_{y \in Y} f(x, y), \\ \\ \text{s.t.: } g(x, y) \leq 0. \end{cases}$$

We agree that whenever $m_1 = 0$ or $m_2 = 0$, this means that the corresponding inequality constraint is absent in the $(BLPP)$. We interpret the $(BLPP)$ as follows.

The problem is divided into two linked subproblems (P_1) and (P_2), where the first one is controlled by the "leader" and the second one by the "follower." The leader has the advantage to fix his strategy $x \in X$ first, then the follower will respond by one strategy $y = y(x)$ or perhaps more, i.e., $y \in O(x)$, where $O(x)$ denotes the set of all optimal solutions to (P_2) for fixed x, in a way such that his objective function is optimized. Following the reaction of the follower, the leader will choose among all admissible $x's$ the one(s) which realize the optimum of his objective function $F(x, y(x))$ if it is possible. We will see in this section the exact definition of an optimal solution to this problem. We note that in the definition we gave, we have supposed that there is some sort of cooperation between the leader and the follower, in the sense that the leader knows all optimal strategies fixed by the follower. This kind of problem will called "cooperatif problem." Some basic definitions are required in the sequel and are given below.

Definition 2.1 (1) *Constraint region of the $(BLPP)$:*

$$S = \{(x, y) \in X \times Y : \quad g(x, y) \leq 0, G(x, y) \leq 0\}.$$

(2) *Feasible set for the second level problem P_2 for each fixed x in X :*

$$S(x) = \{y \in Y : \quad g(x, y) \leq 0\}.$$

(3) *The projection of S on R^{n_1} :*

$$P = \{x \in X : \quad \exists y \in Y \quad s.t. \quad g(x, y) \leq 0, G(x, y) \leq 0\}.$$

(4) *The rational reaction set of (P_2) for $x \in P$:*

$$O(x) = \{y \in S(x) : \quad y \in \arg \min_{z \in S(x)} f(x, z)\}.$$

(5) *The inducible set for the $(BLPP)$:*

$$\overline{S} = \{(x, y) : \quad (x, y) \in S, y \in O(x)\}.$$

$$V(x) := \inf\{P(x)\} = \begin{cases} \inf\limits_{y \in S(x)} f(x,y) & \text{if } S(x) \neq \emptyset, \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $V(\cdot)$ may take the value $-\infty$, because the problem $P(x)$ may not have a solution. The value $+\infty$ is assigned to $V(\cdot)$ by the convention that the infimum over the empty set is equal to $+\infty$. In general, the function $V(\cdot)$ is not differentiable neither convex or Lipschitz or continuous even if the functions F, f, G, g are. In spite of this eventual bad behavior of $V(\cdot)$, this function enables us to reformulate the $(BLPP)$ as a single programming problem. Actually, we have the following result.

Lemma 3.1 *As long as $(BLPP)$ admits a solution, then $(\bar{x}, \bar{y})$ is a solution to the $(BLPP)$ iff $(\bar{x}, \bar{y})$ is a solution to the following problem:*

$$(RBLPP) \qquad \min_{x,y} F(x,y),$$

$$\text{s.t.:} \begin{cases} (x,y) \in X \times Y, \\ f(x,y) - V(x) = 0, \\ G(x,y) \leq 0, \\ g(x,y) \leq 0. \end{cases} \qquad \blacksquare$$

Remark 3.1 (1) Since by the defintion of $V(\cdot)$, we have always $f(x,y) - V(x) \geq 0$, for all (x,y) satisfying $y \in S(x)$ and $x \in X$, the equality constraint in $(RBLPP)$, $f(x,y) - V(x) = 0$ is in fact equivalent to the inequality constraint $f(x,y) - V(x) \leq 0$ whenver $y \in S(x), x \in X$.

(2) We should note that it is implicitely assumed in the lemma that the $(BLPP)$ is cooperatif in the following sense. In the original formulation of the problem,

for each x fixed by the leader, the follower may have more than one choice $y = y(x)$ (i.e., $O(x)$ is not a singleton) and none of those can realize the leader's optimality. The cooperatif assumption results from the constraint $f(x, y) - V(x) = 0$, since this one reflects the fact that the leader knows all vectors $y = y(x)$, solutions of the second level problem $P(x)$ for each fixed x in X. In general, when this cooperatif hypothesis is not assumed, every solution to the $(BLPP)$ is a solution to the $(RBLPP)$, but the inverse is false (see example 5.2). Nevertheless, if for each x in $X, O(x)$ is at most reduced to a singleton, then the problems $(BLPP)$ and $(RBLPP)$ are equivalent.

(3) The set $O(x)$ of definition 2.1 (4), can be rewritten as

$$O(x) = \{y \in S(x) : \ f(x, y) \leq V(x)\}.$$

It can be readily noted that when $m_2 = 0$, then $\overline{S} = Gr(O(\cdot))$, the graph of the multifunction $O(\cdot)$.

From now on, we suppose that the $(BLPP)$ is cooperatif in the sense explained above. Therefore, we will not make any difference between problems $(BLPP)$ and $(RBLPP)$ except if we mention the contrary.

The reformulation of the $(MLPP)$ is given in section 6, where an example of [5] is treated following this reformulation.

In the existence theory studied in section 5, we will need the l.s.c. (lower semi-continuity) of the value function $V(\cdot)$. In the result below, we give sufficient conditions to garantee this property. For this end, we suppose the following.

(h1) $f(\cdot, \cdot), g(\cdot, \cdot)$ are l.s.c. on $X \times Y$.

(h2) There exists $\varepsilon_0 > 0$ such that for each $x \in X$, the following level set

$$B_x(\varepsilon_0) := \{y \in Y : f(x, y) < V(x) + \varepsilon_0\}$$

is included in a fixed compact subset A of Y.

Note that hypothesis **(h2)** can be realized if one of the following hypotheses is satisfied.

(h2.1) Y is compact.

(h2.2) Y is closed and the level set of **(h2)** is uniformly bounded relatively to x, i.e., $\exists M > 0$ s.t. $\sup_{y \in B_x(\varepsilon_0), x \in X} |y| \leq M$.

Lemma 3.2 *Suppose that hypotheses* **(h1)**, **(h2)** *are satisfied. Then we have*

(a) $S(x) \neq \emptyset$ $(\Leftrightarrow V(x) < +\infty) \Rightarrow O(x) \neq \emptyset$.

(b) $V(\cdot)$ *is l.s.c. on* X.

Proof. (a) Let (y_i) be a minimizing sequence for $P(x)$. (Such a sequence exists because of the hypothesis $S(x) \neq \emptyset$.) So for all $i, y_i \in Y, g(x, y_i) \leq 0$ and $\lim_{i \to +\infty} f(x, y_i) = V(x)$. Therefore, for i big enaugh, $f(x, y_i) < V(x) + \varepsilon_0$, i.e., $y_i \in B_x(\varepsilon_0)$. By hypothesis **(h2)**, we can suppose without loss of generality that (y_i) converges to some $y \in Y$. The l.s.c. of f and g implies that $g(x, y) \leq 0$ and $V(x) \geq f(x, y)$. Consequently, y is feasible for $P(x)$ and then $f(x, y) \geq V(x)$. Hence $V(x) = f(x, y)$ and $y \in O(x)$.

(b) Let $(x_i) \subset X$ be any sequence converging to $x \in X$. We have to prove that $\liminf_{i \to +\infty} V(x_i) \geq V(x)$. We can suppose without loss of generality that $\liminf_{i \to +\infty} V(x_i) = \lim_{i \to +\infty} V(x_i) < \infty$. Let $y_i \in S(x_i)$ s.t. $f(x_i, y_i) = V(x_i)$ (at least after certain rank for i). Since $f(x_i, y_i) < V(x_i) + \varepsilon_0 \;\; \forall\, i$, then by **(h2)**, we can extract a subsequence which we don't relabel converging to some $y \in Y$. Using the l.s.c. of g, we have $y \in S(x)$; hence $V(x) \leq f(x, y)$. Using the l.s.c. of f, we deduce that $f(x, y) \leq \liminf_{i \to +\infty} V(x_i)$. Consequently, $V(x) \leq \liminf_{i \to +\infty} V(x_i)$. ∎

In the following lemma we keep hypothesis **(h1)** in force and we replace hypothesis **(h2)** by the weaker one.

(h2)' There exists $\varepsilon_0 > 0$ and $\bar{x} \in X$ s.t. $V(\bar{x}) < \infty$ and the following set:

$$B'_x(\varepsilon_0, \bar{x}) := \{y \in Y : \; f(x,y) < V(\bar{x}) + \varepsilon_0\}$$

is included in a fixed compact set A of Y for all $x \in X$.

Note that this hypothesis can be satisfied if for example we can find an $\bar{x} \in X$ for which we have

(i) $B'_{\bar{x}}(\varepsilon_0, \bar{x}) \subset A \subset Y$, with A being compact,

(ii) $\bar{x} \in \arg\min_{x \in X} f(x,y)$ for each fixed $y \in Y$.

Lemma 3.3 *Under hypotheses* **(h1)** *and* **(h2)'**, *we have*

$$0 \leq \varepsilon < \varepsilon_0 \quad \text{and} \quad V(x) \leq V(\bar{x}) + \varepsilon \Rightarrow O(x) \neq \emptyset.$$

Proof. The proof is quite similar to the one of the previous lemma and thus is omitted. ■

In the following proposition we give some properties of the multifunction $O(\cdot)$: $R^{n_1} \to 2^{R^{n_2}}, x \mapsto O(x)$. Of course we suppose that for all x in $X, S(x) \neq \emptyset$. This implies according to Lemma 3.2 (a) that $O(x) \neq \emptyset \;\; \forall x \in X$.

Proposition 3.1 *Suppose that hypotheses* **(h1)**, **(h2)** *are satisfied and that* $V(\cdot)$ *is u.s.c. Then*

(i) *The multifunction* $O(\cdot)$ *is closed, i.e.,*

$$\left.\begin{array}{l} y_i \in O(x_i) \\ x_i \to x \\ y_i \to y \end{array}\right\} \Rightarrow \quad y \in O(x).$$

(ii) $\forall x \in X, O(x)$ *is compact.*

Proof. (i) The relation $y_i \in O(x_i)$ is equivalent to say that $y_i \in S(x_i)$ and $f(x_i, y_i) = V(x_i)$ for all i. Since $f(x_i, y_i) = V(x_i) < V(x_i) + \varepsilon_0$ and $y_i \to y$, then by **(h2)**, we must have $y \in Y$. Now the l.s.c. of $g(\cdot,\cdot)$ implies that $g(x,y) \leq 0$. So $y \in S(x)$. Finally, we have

$$\begin{aligned} f(x,y) &\leq \liminf_{i\to+\infty} f(x_i, y_i) \quad (f \text{ is l.s.c.}) \\ &= \liminf_{i\to+\infty} V(x_i) \\ &\leq \limsup_{i\to+\infty} V(x_i) \qquad (V \text{ is u.s.c.}) \\ &\leq V(x). \end{aligned}$$

This implies that $y \in O(x)$ and the proof of part (i) is finished. Part (ii) is a consequence of (i). ■

4 Non-convexity of the $(BLPP)$

Before we discuss the nonconvex nature of the $(BLPP)$, we give the following definition.

Definition 4.1 *Let $X \subset R^{n_1}, Y \subset R^{n_2}$ be two nonempty convex sets. Let $M : X \to 2^Y$ be a multifunction. Then we say that $M(\cdot)$ is convex if it satisfies the following conditions:*

$$\left.\begin{array}{l} y_1 \in M(x_1) \\ y_2 \in M(x_2) \end{array}\right\} \Rightarrow \lambda y_1 + (1-\lambda) y_2 \in M(\lambda x_1 + (1-\lambda) x_2)$$

$\forall\ \lambda \in [0,1]$ and $\forall\ x_1, x_2 \in X$. That is

$$\lambda M(x_1) + (1-\lambda) M(x_2) \subset M(\lambda x_1 + (1-\lambda) x_2) \qquad \forall\ x_1, x_2 \in X, \forall\ \lambda \in [0,1].$$

Note that when $M(\cdot)$ is a point-to-point function, the definition given above coincides with the usual definition of an affine function.

Perhaps the big difficulty to solve the $(BLPP)$ is its nonconvex aspect as was shown in many papers, [1], [2], [3], [4], [6], [7], [15] and others. In fact, the definition of an optimal solution for this problem indicates that this last one is convex in the standard sense if the function $F(\cdot,\cdot)$ is convex and the inducible set $\overline{S}$ is convex too. The convexity of $\overline{S}$ is related not only to that of the functions $f(\cdot,\cdot), g(\cdot,\cdot)$ and $G(\cdot,\cdot)$ and the sets X, Y, but also to the convexity of the multifunction $O(\cdot)$. Moreover, the last requirement cannot be garanteed even when the second level is linear as was shown in many papers. Since $O(\cdot)$ is linked to $f(\cdot,\cdot), V(\cdot)$ and $S(\cdot)$ (so to Y and $g(\cdot,\cdot)$), then even when $f(\cdot,\cdot)$ and $-V(\cdot)$ are convex, $S(\cdot)$ is convex (as a multifunction), $O(\cdot)$ may be nonconvex and a fortiori $\overline{S}$ (see example 5.2). So in order to assure the convexity of $\overline{S}$, we should put a strong hypothesis on the function $V(\cdot)$, which guarantees the convexity of the multifunction $O(\cdot)$. In fact, we have the following result.

Proposition 4.1 *(a) Suppose that for each fixed x in X, the functions $f(x,\cdot)$, $g(x,\cdot)$ are convex and the set Y is convex. Then $O(x)$ is a convex set.*

(b) Suppose that $f(\cdot,\cdot), g(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are convex functions, X, Y are convex sets and $O(\cdot)$ is convex (in particular this is the case if $V(\cdot)$ is affine for example). Then $\overline{S}$ is convex.

Poof. (a) The proof of this part is straightforward and thus is omitted.

(b) Recall that $\overline{S} = S \cap \{(x,y) : \ y \in S(x), y \in O(x)\}$. It is obvious that hypotheses imply that S is convex. The convexity of $g(\cdot,\cdot)$ and Y implies that the multifunction $S(\cdot)$ is convex. So it remains just to prove the convexity of the multifunction $O(\cdot)$ when $V(\cdot)$ is supposed to be affine. Indeed, let $\lambda \in [0,1], y_1 \in O(x_1), y_2 \in O(x_2)$, with $x_1, x_2 \in X$. Then

$$
\begin{aligned}
f\Big(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\Big) &\leq \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2) \\
&\quad (f(\cdot,\cdot) \text{ is convex}) \\
&\leq \lambda V(x_1) + (1-\lambda) V(x_2) \\
&\quad (\text{because } y_i \in O(x_i), i = 1, 2) \\
&= V\Big(\lambda x_1 + (1-\lambda)x_2\Big) \\
&\quad (\text{since } V(\cdot) \text{ is affine}).
\end{aligned}
$$

This implies that $\lambda y_1 + (1-\lambda)y_2 \in O(\lambda x_1 + (1-\lambda)x_2)$, and the proof is achieved.

■

Remark 4.1 The hypothesis that "$V(\cdot)$ is affine" cannot be weakned. Even if $V(\cdot)$ is piecewise affine as when the second level is linear, $\overline{S}$ is not necessarily convex. In the linear case for example, $V(\cdot)$ is piecewise affine, however $\overline{S}$ can be formed of more than one face of the polydron generating the set of constraints S. It seems by examining the last inequalities sequence in the proof above that the concavity of $V(\cdot)$ suffices to garantee the convexity of $\overline{S}$. However, it is possible that for some $\lambda \in [0,1]$ and $x_1, x_2 \in X$, $\lambda V(x_1) + (1-\lambda)V(x_2) < V(\lambda x_1 + (1-\lambda)x_2)$. Example 5.2 provides a situation where all hypotheses of part (b) are satisfied, $V(\cdot)$ is concave, yet $\overline{S}$ is not convex. In conclusion, the class of $(BLPP)$ problems which are convex, is very small and we should think of the $(BLPP)$ as a heighly difficult problem. Note that the hypothesis that $V(\cdot)$ is affine is only a sufficient condition for $O(\cdot)$ to be convex and not a necessary condition; an example can be provided easily.

As an immediate consequence of the proposition above is the following.

Proposition 4.2 *Suppose that hypotheses of the preceding proposition (b) are satisfied and in addition that $F(\cdot,\cdot)$ is convex. Then if a solution to the $(BLPP)$ exists, it is a gblobal one.*

The counterexample given in [9] can be seen as an application of this last proposition, where $\overline{S}$ is formed of a unique face of the polydron generating S, the reason for which $V(\cdot)$ is affine.

5 Existence

Existence is a step whose role is to reassure us before attacking different tools in order to identify solutions (sufficient conditions) or at least possible candidats (necessary conditions). In this paper, we limit ourselves to existence theory. In general, the $(BLPP)$ does not admit a solution. As an example, we give the following.

Example 5.1 We consider the following problem:

$$(P) \quad \min_{-1\leq x\leq 1}(x-y),$$

where for each fixed $x, y = y(x)$ is a solution to the problem:

$$\min_{y\geq -1} xy,$$

s.t.: $y \geq -x-1$.

So the data of (P) in the standard form of $(BLPP)$ are: $F(x,y) = x-y, f(x,y) = xy, G(x,y) = -x-y-1, X = [-1,1]$ and $Y = [-1,+\infty[$. For this problem we have

$$S = \{(x,y): \ -1 \leq x < 0, y \geq -1-x\} \cup \{(x,y): \ 0 \leq x \leq 1, y \geq -1\},$$

a set which is unbounded and closed (see fig.1).

The set of solutions to the second level problem for each fixed x in X is given by

$$O(x) = \begin{cases} \emptyset & : -1 \leq x < 0, \\ [-1,+\infty[& : x = 0, \\ -1 & : 0 < x \leq 1. \end{cases}$$

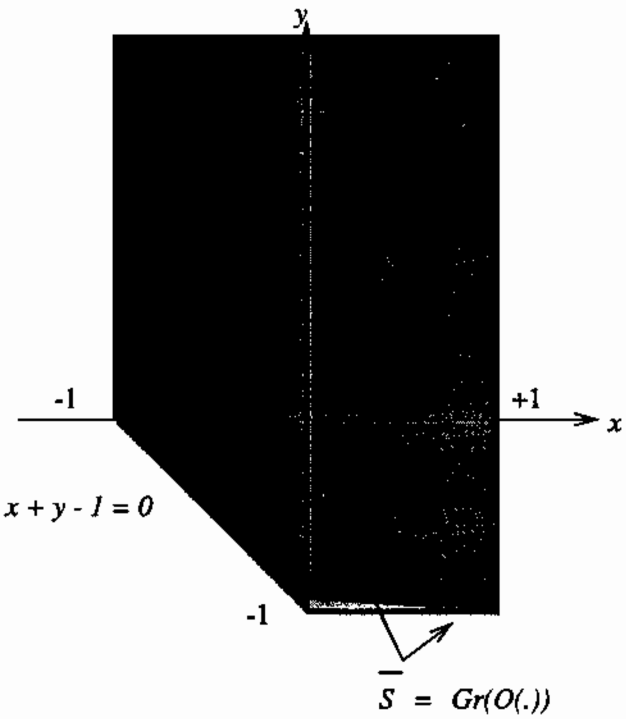

Figure 1: The geometry of Example 5.1

So the domain of the multifunction $O(\cdot)$ is $[0,1]$ (its graph is shown in fig.1).

Consequently, the value function $V(\cdot)$ corresponding to the second level is given by

$$V(x) = \begin{cases} -\infty & : -1 \leq x < 0, \\ 0 & : x = 0, \\ -x & : 0 < x \leq 1. \end{cases}$$

Thus $V(\cdot)$ is u.s.c. at 0 and is not l.s.c. at the same point. Moreover, $V(\cdot)$ is finite only on the interval $[0,1]$.

Let examine now if the problem has a solution. If it is the case, then the corresponding $(RBLPP)$ must have a solution (Lemma 2.1) and the formulation of this last one is

$$
(P') \quad \min_{x,y}(x-y)
$$

$$
\text{s.t.:}\begin{cases} -1 \le x \le 1, & (1)\\ y \ge -1, & (2)\\ y \ge -x-1, & (3)\\ g(x,y) \le 0, & (4)\end{cases}
$$

where
$$
g(x,y) := f(x,y) - V(x) = \begin{cases} +\infty & : -1 \le x < 0, y \ge -x-1,\\ 0 & : x = 0, y \ge -1,\\ xy + x & : 0 < x \le 1, y \ge -1.\end{cases}
$$

Therefore when $0 < x \le 1, g(x,y) \le 0 \Leftrightarrow y = -1$. Let define the following sets:

$$
D_1 := \{(x,y) \in R^2 : x = 0, y \ge -1\},
$$
$$
D_2 := \{(x,y) \in R^2 : 0 < x \le 1, y = -1\}.
$$

Then D_1 and D_2 are disjoint sets and (x,y) is feasible for (P') iff $(x,y) \in D_1 \cup D_2 = \overline{S} = Gr(O(\cdot))$. We have

$$
\inf_{(x,y)\in D_1}(x-y) = \inf_{x=0,y\ge -1}(x-y) = -\infty,
$$
$$
\inf_{(x,y)\in D_2}(x-y) = \inf_{0<x\le 1}(x+1) \quad \text{(this infimum cannot be attained).}
$$

In conclusion, (P) does not have any solution in both cases cooperatif case and noncooperatif case. ■

Throughout this section, we assume the following hypotheses:

(H1) $F(\cdot,\cdot), f(\cdot,\cdot), g(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are l.s.c. on $X \times Y$.

(H2) $\inf(BLPP) < \infty, V(\cdot)$ is u.s.c. on X.

(H3) There exists $\varepsilon_0 > 0$ s.t. the set

$$B(\varepsilon_0) = \{(x, y) \in S : F(x, y) < \inf(BLPP) + \varepsilon_0\},$$

is included in a fixed compact subset B of $X \times Y$.

Let us note that hypothesis **(H3)** is satisfied if for example $X \times Y$ is compact or if $X \times Y$ is closed and $B(\varepsilon_0)$ is bounded.

Proposition 5.1 *Under the hypotheses* **(H1)-(H3)***, the* $(BLPP)$ *has at least one optimal solution.*

Proof. Let $\{(x_i, y_i)\}$ be a minimizing sequence for the $(BLPP)$, i.e., $f(x_i, y_i) - V(x_i) = 0, G(x_i, y_i) \leq 0, g(x_i, y_i) \leq 0, (x_i, y_i) \in X \times Y \;\; \forall\, i$ and $\lim_{i \to +\infty} F(x_i, y_i) = \inf(BLPP)$. So for i big enough, $F(x_i, y_i) < \inf(BLPP) + \varepsilon_0$, i.e., $(x_i, y_i) \in B(\varepsilon_0)$. Now by **(H3)**, we can pass to a subsequence if necessary to confirm that $\{(x_i, y_i)\}$ converges to some $(x, y) \in X \times Y$. The l.s.c. of $f(\cdot, \cdot)$ and the u.s.c. of $V(\cdot)$ imply that $f(x, y) - V(x) \leq \liminf_{i \to +\infty} f(x_i, y_i) - \limsup_{i \to +\infty} V(x_i) = \liminf_{i \to +\infty} \{f(x_i, y_i) - V(x_i)\} \leq 0$. Since $F(\cdot, \cdot), g(\cdot, \cdot)$ and $(G(\cdot, \cdot)$ are l.s.c., then it follows that (x, y) is feasible for the $(BLPP)$ and $\liminf_{i \to +\infty} F(x_i, y_i) \geq F(x, y)$. Consequently, $F(x, y) = \inf(BLPP)$ and (x, y) is an optimal solution to the $(BLPP)$. ∎

When the inequality constraint $G \leq 0$ is absent, i.e., $m_2 = 0$, an alternate existence result based on the closedness property of the multifunction $O(\cdot)$ can be derived. The hypothesis **(H3)** will be replaced by the following one.

(H3)' The set $\{(x, y) \in X \times Y : \; f(x, y) \leq V(x)\}$ is compact.

Proposition 5.2 *Suppose that* $m_2 = 0, \inf(BLPP) < \infty$ *and that hypotheses* **(H1)**,**(h2)** *and* **(H3)'** *are satisfied. Then the* $(BLPP)$ *has at least one optimal solution.*

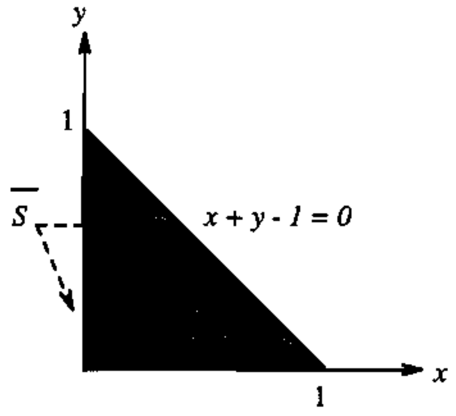

Figure 2: The geometry of Example 5.2

Proof. Since the constraint $G \leq 0$ is absent, then according to remark 3.1 (3), one has $\overline{S} = Gr(O(\cdot))$. Now according to Prop.3.1 (i), the multifunction $O(\cdot)$ is closed. So the $(BLPP)$ is equivalent to the following problem:

$$\min_{(x,y)\in Gr(O(\cdot))} F(x,y).$$

Then using l.s.c. of $F(\cdot,\cdot)$, **(H3)'** and the closedness of $O(\cdot)$ we deduce that this last problem has at least one optimal solution and so the $(BLPP)$ also. ∎

As we have mentioned in remark 3.1 (2), a $(BLPP)$ can have a solution in the cooperatif case and the same problem can have no solution in the noncooperatif case. To show this, we give the following example.

Example 5.2 Let us consider the following problem:

$(P)\quad \min_{x\geq 0}(x+y),$

s.t. for each fixed $x, y = y(x)$ is a solution to the problem :

$\max_{y\geq 0} xy$

s.t.: $x + y - 1 \leq 0.$

The region of constraints S corresponding to this problem is shown in fig.2.

It is clear that existence hypotheses **(H1)**-**(H3)** are satisfied. So (P) has a solution in the cooperatif case of course.

Let x be fixed. Then the value function $V(\cdot)$ of the second level problem is

$$V(x) = x\max\{y : 0 \leq y \leq 1 - x\} = \begin{cases} 0 & if \;\; x = 0, \\ x(1-x) & if \;\; x \in]0,1]. \end{cases}$$

That is, for $0 < x \leq 1$, the problem defined by $V(\cdot)$ has a unique solution given by $\bar{y}(x) = 1 - x$ and if $x = 0$, then for every choice $y \in [0,1]$ by the follower, its objective function takes the value 0. It happens in this example that $V(\cdot)$ is continuously differentiable everywhere in $[0,1]$. However it is not always the case (for an example, see Exemple 3.1 of [4] or [19]).

Consequently, the inducible set for this problem is

$$\overline{S} = \{(x, 1-x) : 0 < x \leq 1\} \cup \{(0,y) : 0 \leq y \leq 1\},$$

a nonconvex closed set ($\overline{S}$ is indicated in bold in fig.2). As this problem does not contain a coupled inequality constraint in the first level, then according to remark 3.1 (3), we have $Gr(O(\cdot)) = \overline{S}$. So

$$(P) \quad \Leftrightarrow \quad \min_{x,y} F(x,y),$$

$$\text{s.t.:} \begin{cases} x + y - 1 \leq 0, \\ -x < 0, \\ -y \leq 0, \end{cases} \qquad \text{or} \qquad \begin{cases} y - 1 \leq 0, \\ -y \leq 0, \\ x = 0, \end{cases}$$

where

$$F(x,y) = \begin{cases} y & \text{if } \; x = 0, 0 \leq y \leq 1, \\ x + \bar{y}(x) = 1 & \text{if } \; 0 < x \leq 1, 0 \leq y \leq 1. \end{cases}$$

Suppose that the problem is cooperatif. Note that when $0 < x \leq 1$, $F(x, y(x))$ is identically equal to 1 and when $x = 0$, the minimum value of $F(x, y(x))$ is attained at $y = 0$ and is equal to 0. Consequently, the unique solution to this problem in the cooperatif case is the point $(0, 0)$.

Suppose now that the problem is not cooperatif. Then when $x = 0$, the follower does not realize any gain and he can be indifferent with the leader by choosing $y \in]0, 1]$, a choice which does not allow to the leader to attain his objective. Whereas if $x \in]0, 1]$, then the follower would have prefered that the leader choose $x = 1/2$ in order to attain his optimized objective of $1/4$, a choice for which the leader will not realize his infimum of 0. In conclusion, (P) does not have a solution in the noncooperatif case.

We leave to the reader to check that the reformulation of (P) according to Lemma 3.1, will give the same conclusion as in the coooperatif case. ■

6 General multi-level programming problem

Let $p \geq 1, n_p \geq 1, m_p \geq 0$ be given integers. Let $F^i : R^{n_1} \times \ldots \times R^{n_p} \to R$ $(i = 1, \ldots, p), G^i : R^{n_1} \times \ldots \times R^{n_p} \to R^{m_i}$ $(i = 1, \ldots, p)$ be given functions and $X^1, \ldots, X^p$ be given subsets of $R^{n_1}, \ldots, R^{n_p}$ respectively. We set $G^i = [G^i_1, \ldots, G^i_{m_i}]$ $(i = 1, \ldots, p)$. A generic vector from $R^{n_1} \times \ldots \times R^{n_p}$ is denoted by $(x^1, \ldots, x^p)$.

The multi-level programming problem can be stated as follows:

$$
(MLPP) \qquad (P_1) \quad \begin{cases} \min\limits_{x^1 \in X^1} F^1(x^1, \dots, x^p), \\ \text{s.t.} \quad G^1(x^1, \dots, x^p) \le 0, \end{cases}
$$

s.t. for each fixed $x^1, x^2 = x^2(x^1)$ solves

$$
(P_2) \quad \begin{cases} \min\limits_{x^2 \in X^2} F^2(x^1, x^2), \\ \text{s.t.} \quad G^2(x^1, x^2) \le 0, \end{cases}
$$

s.t. for each fixed $x^1, x^2, x^3 = x^3(x^1, x^2)$ solves

$$
(P_3) \quad \begin{cases} \min\limits_{x^3 \in X^3} F^3(x^1, x^2, x^3), \\ \text{s.t.} \quad G^3(x^1, x^2, x^3) \le 0, \end{cases}
$$

....

....

....

s.t. for each fixed $x^1, \dots, x^{p-2}, x^{p-1} = x^{p-1}(x^1, \dots, x^{p-2})$ solves

$$
(P_{p-1}) \quad \begin{cases} \min\limits_{x^{p-1} \in X^{p-1}} F^{p-1}(x^1, \dots, x^{p-1}), \\ \text{s.t.} \quad G^{p-1}(x^1, \dots, x^{p-1}) \le 0, \end{cases}
$$

s.t. for each fixed $x^1, \dots, x^{p-1}, x^p = x^p(x^1, \dots, x^{p-1})$ solves

$$
(P_p) \quad \begin{cases} \min\limits_{x^p \in X^p} F^p(x^1, \dots, x^p), \\ \text{s.t.} \quad G^p(x^1, \dots, x^p) \le 0. \end{cases}
$$

We agree that whenever $m_i = 0$, this means that the corresponding inequality constraint G^i is absent.

The following definition corresponds to Def. 2.1.

Definition 6.1 (1) *The region of constraints for $(MLPP)$:*

$$
S =: \{x = (x^1, \dots, x^p) \in X^1 \times \dots \times X^p : \; G^i(x^1, \dots, x^i) \le 0 \;\; (i = 1, \dots, p)\}.
$$

(2) *Feasible set for the i^{th} $(i = 2, \dots, p)$ level of $(MLPP)$:*

$$
S(x^1, \dots, x^{i-1}) := \{x^i \in X^i : \; G^i(x^1, \dots, x^i) \le 0\}.
$$

(3) *The reaction set of* (P_i) *for fixed* $(x^1,\dots .x^{i-1}), (i=2,\dots,p)$

$$O(x^1,\dots,x^{i-1}) := \{\bar{x}^i \in S(x^1,\dots,x^{i-1}) :\ \bar{x}^i \in \arg \min_{x^i \in S(x^1,\dots,x^{i-1})} F^i(x^1,\dots,x^i)\}.$$

(4) *The inducible set for* $(MLPP)$:

$$\overline{S} := \{(x^1,\dots,x^p) \in S :\ x^i \in O(x^1,\dots,x^{i-1}) \quad (i=2,\dots,p)\}.$$

Let us define the value functions corresponding to each level of the $(MLPP)$, $V^i : R^{n_1} \times \ldots \times R^{n_p} \to R \cup \{+\infty, -\infty\} \quad (i=2,\dots,p)$ s.t.

$$V^i(x^1,\dots,x^{i-1}) := \inf_{x^i \in S(x^1,\dots,x^{i-1})} F^i(x^1,\dots,x^{i-1}),$$

with the convention that the infimum over the empty set is equal to $+\infty$. Thus, the following lemma corresponds to lemma 3.1.

Lemma 6.1 *As long as the* $(MLPP)$ *has a solution, then the vector* $(\bar{x}^1,\dots,\bar{x}^p)$ *is a solution to the* $(MLPP)$ *iff it is a solution to the following problem*:

$$(RMLPP) \qquad \min_{(x^1,\dots,x^p)} F^1(x^1,\dots,x^p),$$

$$s.t. \begin{cases} F^i(x^1,\dots,x^i) - V^i(x^1,\dots,x^{i-1}) = 0 \quad (i=2,\dots,p), \\ G^i(x^1,\dots,x^i) \le 0 \quad (i=1,\dots,p), \\ (x^1,\dots,x^p) \in X^1 \times \ldots \times X^p. \end{cases}$$

As an example we consider the same problem suggested in [5].

Example 6.1 The problem is the following:

$$
(P) \qquad \begin{aligned}
&\min_{0\le x_1\le 6}(-x_1+x_2-x_3),\\
&\text{s.t. for given } x_1, x_2 = x_2(x_1) \text{ solves:}\\
&\min_{x_2\ge 0} -x_2,\\
&\text{s.t.}\begin{cases} x_1 \ge 0,\\ 2x_1+x_2-10\le 0,\end{cases}\\
&\text{s.t. for given } x_1, x_2, x_3 = x_3(x_1,x_2) \text{ solves:}\\
&\min_{x_3\ge 0} -x_3,\\
&\text{s.t.}\begin{cases} x_1 \ge 0,\\ x_2\ge 0,\\ 2x_1+x_2+x_3-18\le 0.\end{cases}
\end{aligned}
$$

Thus, the data of this problem are:

$p = 3; n_1 = n_2 = n_3 = 1; m_1 = 0, m_2 = 2$ and $m_3 = 3$.

$F^1(x_1, x_2, x_3) = -x_1 + x_2 - x_3$.

$F^2(x_2) = -x_2$.

$F^3(x_3) = -x_3$.

$G^2(x_1, x_2) = [2x_1 + x_2 - 10, -x_1]$.

$G^3(x_1, x_2, x_3) = [2x_1 + x_2 + x_3 - 18, -x_1, -x_2]$.

$X^1 = [0, 6]$.

$X^2 = [0, +\infty[$.

$X^3 = [0, +\infty[$.

It is obvious that we have

$$V^3(x_1, x_2) = 2x_1 + x_2 - 18,$$

$$V^2(x_1) = 2x_1 - 10.$$

Therefore, the reformulation of (P) according to the previous lemma leads to the following problem:

$$(P') \qquad \min_{(x_1,x_2,x_3)} (-x_1 + x_2 - x_3),$$

$$\text{s.t.} \begin{cases} -x_3 - 2x_1 - x_2 + 18 = 0, & (1) \\ -x_2 - 2x_1 + 10 = 0, & (2) \\ 2x_1 + x_2 - 10 \le 0, & (3) \\ 2x_1 + x_2 + x_3 - 18 \le 0, & (4) \\ 0 \le x_1 \le 6, & (5) \\ x_2 \ge 0, & (6) \\ x_3 \ge 0. & (7) \end{cases}$$

Let (x_1, x_2, x_3) be a solution to problem (P'). Then from (1) and (2) it follows that $x_3 = 8$ necessarily. Thus (3) and (4) are identical and using (5), (6) we deduce that (P') is equivalent to the minimization of $-3x_1$ on the interval $[0, 5]$, which has the unique solution $x_1 = 5$; hence $x_2 = 0$ by (2) or (1). Consequently, the point $(5, 0, 8)$ is the unique solution to (P) as was pointed out in [5]. The geometry of this example is shown in fig.3. ∎

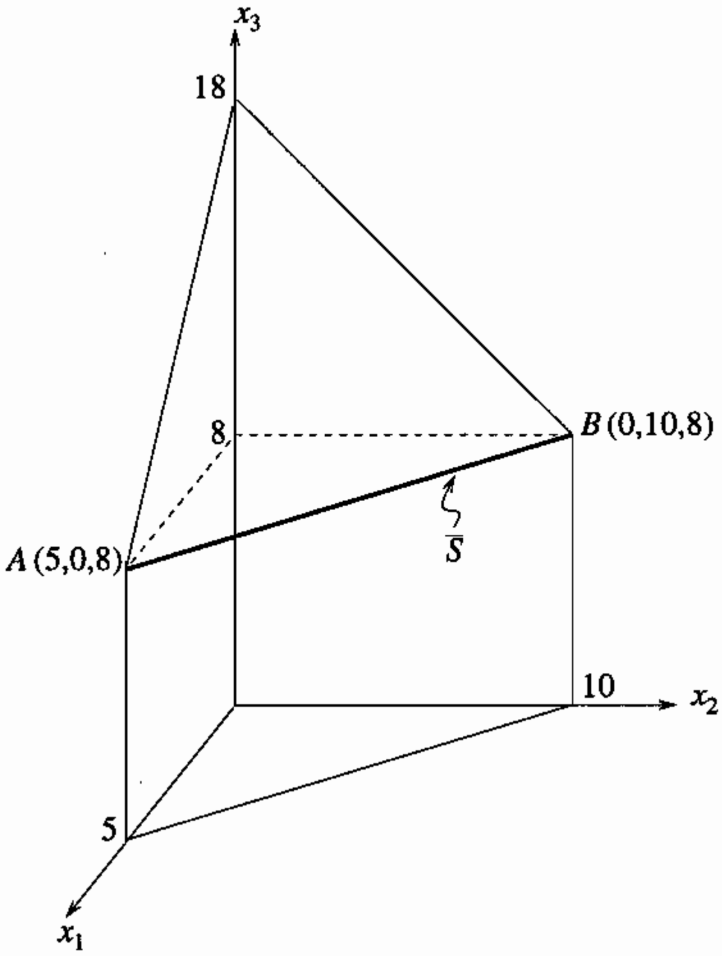

Figure 3: The geometry of Example 6.1

7 Application

As an application of the $(BLPP)$ we consider the model defined below based on the continuous Lanchester model [14]. This application constitutes a small part of a work which is now under preparation with professors M. Breton and G. Zaccour.

The model: We suppose we have two competitors in a competition for a market share (let say M for the leader and $1-M$ for the follower). Each of the two players wishes to maximize its discounted profit:

$$\text{Leader's objective:} \quad \max_{A} J^1 = \sum_{t=0}^{T} \rho^t r_1(A_t, M_t)$$

$$\text{Follower's objective:} \quad \max_{B} J^2 = \sum_{t=0}^{T} \rho^t r_2(B_t, M_t)$$

Where

$$r_1(B_t, M_t) = g_1 M_t - A_t$$

$$r_2(B_t, M_t) = g_2(1 - M_t) - B_t$$

The leader's market share M varies over $t = 0, 1, 2, \ldots, T$ via the equation

$$M_{t+1} = M_t + \left[\beta_1 A_t^{\alpha_1}(1 - M_t) - \beta_2 B_t^{\alpha_2} M_t\right] \quad t = 0, 1, \ldots, T,$$

with M_0, the initial market share is given. We explain below the meaning of each element present in the model.

A_t	The leader's advertising level at period t.
B_t	The follower's advertising level at period t.
$0 \leq \rho \leq 1$	The discount factor.
$g_1 > 0$	The leader's market share value parameter.

$g_2 > 0$	The follower's market share value parameter.
$0 < \alpha_1 < 1$	The leader's elasticity parameter.
$0 < \alpha_2 < 1$	The follower's elasticity parameter.
$\beta_1 > 0$	The leader's advertising effectiveness parameter.
$\beta_2 > 0$	The follower's advertising effectiveness parameter.

This is a dynamic bi-level programing problem. However, in order to solve it, we will use static bi-level programming as we will see below.

Solution: We divide the problem into $T+1$ periods. At period $i \in [1 \ldots T+1]$ the leader's and follower's objective functions are as follows:

$$J_i^1 = \sum_{t=0}^{i} \rho^t r_1(A_t, M_t),$$

$$J_i^2 = \sum_{t=0}^{i} \rho^t r_2(B_t, M_t),$$

where functions r_1 and r_2 are extended in a natural way to period $T+1$, and where of course we take $A_{T+1} = B_{T+1} = 0$ at the last period since at this period the game is over and the advertising are unnecessary. Therfore we have $M_{T+2} = M_{T+1}$. In order to solve the problem globally, we will proceed by solving it period by period, where at each period we have a static bilevel problem. The solution we will give is a closed-loop solution. That is advertising functions depend only on market share. The case where advertising functions deppend only on time correspond to the open-loop solution. The advantage of the present solution is that it has the same pattern as the open-loop solution of [14]. But the most interesting thing is the fact that the present one is explicit and no numerical computations are needed. The solution procedure is as follows.

Period 1 For fixed A_0, the follower's problem is the following:

$$\max_{B_0} J_0^2 = \sum_{t=0}^{1} \rho^t r_2(B_t, M_t) = \Big[g_2(1-M_0) - B_0\Big] + \rho\Big[g_2(1-M_1) - B_1\Big]$$

with

$$M_1 = M_0 + \beta_1 A_0^{\alpha_1}(1-M_0) - \beta_2 B_0^{\alpha_2} M_0 \tag{1}$$

The solution $\bar{B}_0$ to this problem is such that

$$-1 + \rho\alpha_2\beta_2 g_2 M_0 \bar{B}_0^{\alpha_2 - 1} = 0,$$

i.e.,

$$\bar{B}_0 = \Big[\rho\alpha_2\beta_2 g_2 M_0\Big]^{\frac{1}{1-\alpha_2}}.$$

Considering $B_0 = B_0(A_0)$ via (1), the leader's problem becomes

$$\max_{A_0} J_0^1 = \sum_{t=0}^{1} \rho^t r_1(A_t, M_t) = \Big[g_1(M_0 - A_0\Big] + \rho\Big[g_1 M_1 - A_1\Big]$$

where M_1 is given by (1). The solution to this problem is such that

$$-1 + \rho g_1 \alpha_1 \beta_1 (1-M_0)\bar{A}_0^{\alpha_1 - 1} = 0$$

i.e.,

$$\bar{A}_0 = \Big[\rho g_1 \alpha_1 \beta_1 (1-M_0)\Big]^{\frac{1}{1-\alpha_1}}.$$

Period 2 The same computations presented in period 1 are repeated for the present period to leads to

$$\bar{B}_1 = \Big[\rho\alpha_2\beta_2 g_2 M_1\Big]^{\frac{1}{1-\alpha_2}},$$

and

$$\bar{A}_1 = \Big[\rho g_1 \alpha_1 \beta_1 (1-M_1)\Big]^{\frac{1}{1-\alpha_1}}.$$

The procedure continuous in this way until we arrive to period $T+1$.

Period $T+1$ In this period, for fixed A_T, the follower's objective is

$$\max_{B_T} J^2_{T+1} = \sum_{t=0}^{T+1} r_2(B_t, M_t).$$

By the equation

$$M_{T+1} = M_T + \beta_1 A_T^{\alpha_1}(1-M_T) - \beta_2 B_T^{\alpha_2} M_T \tag{2}$$

it follows that the solution $\bar{B}_T$ to the follower's problem is such that

$$-1 + \rho\alpha_2\beta_2 g_2 M_T \bar{B}_T^{\alpha_2 - 1} = 0,$$

i.e.,

$$\bar{B}_T = \left[\rho\alpha_2\beta_2 g_2 M_T\right]^{\frac{1}{1-\alpha_2}}$$

Considering $B_T = B_T(A_T)$ via (2), it follows that the solution to the leader's problem is given by:

$$\bar{A}_T = \left[\rho g_1 \alpha_1 \beta_1 (1 - M_T)\right]^{\frac{1}{1-\alpha_1}}$$

The conclusion is that the solution to the model can be computed reccursively via the following system:

- $\bar{B}_t = \left[\rho\alpha_2\beta_2 g_2 M_t\right]^{\frac{1}{1-\alpha_2}} \qquad t = 0, 1, 2, \ldots, T$
- $\bar{A}_t = \left[\rho g_1 \alpha_1 \beta_1 (1 - M_t)\right]^{\frac{1}{1-\alpha_1}} \qquad t = 0, 1, 2, \ldots, T$
- $M_t = M_{t-1} + \left[\beta_1 \bar{A}_{t-1}^{\alpha_1}(1 - M_{t-1}) - \beta_2 \bar{B}_{t-1}^{\alpha_2} M_{t-1}\right] \qquad t = 1, 2, \ldots, T$ with M_0 given.

We say that the problem is symetric if corresponding parameters for the leader and the follower are equal. It is easy to prove that in the symetric case, the market share remains constant iff the initial market share is equal to 0.5 for the optimal solution below and in this case the optimal strategies pairs $(\bar{A}_t, \bar{B}_t)$ are equal of course. If M_0 is not necessarily equal to 0.5, we have the following result. A general result for any kind of solution is available but not stated here.

Proposition 7.1 *Suppose we are in the symetric case. The market share converges to 0.5 when the horizon planing T is sufficiently large, and the optimal advertising functions converge to the constant $(\rho\alpha\beta g/2)^{1/(1-\alpha)}$.*

Proof. Let $\alpha_i = \alpha, \beta_i = \beta$ and $g_i = g \quad (i = 1, 2)$. Then for $t = 1, 2, \ldots, T$

$$M_t = M_{t-1} + C\Big[(1 - M_{t-1})^a - M_{t-1}^a\Big] \tag{3}$$

where

$$C = \beta(\rho\alpha\beta g)^{\frac{\alpha}{1-\alpha}}$$

$$a = \frac{1}{1-\alpha} \in]1, +\infty[.$$

Since the sequence $\{M_t\}_t$ must converge at least for a subsequence to some $M \in [0, 1]$, it follows by passing to the limit as $t \to +\infty$ in (3)

$$(1 - M)^a = M^a.$$

Since the numerical function defined on $[0, 1]$ by $f(x) = x^a$ is one to one on $[0, 1]$, the last equation has a unique solution, namely $M = 0.5$ and the proof is acheaved.
■

In figures 4, 5 and 6 we have presented a particular case where the problem is symetric with the following parameter values: $M_0 = 0.25, T = 30, \rho = 1, \alpha_i = 0.5, \beta_i = 0.5$ and $g_i = 1 \quad (i = 1, 2)$. It is not surprising that the market share converges to its steady-state value of 0.5 over a sufficient large horizon T, the same conclusion as that of [14] confirmed here by the same proposition. Advertising functions converge to the same steady-control value which is equal 1/64=0.015625 by the last proposition. The advantage of the present solution is that advertising values are very low with respect to the results addressed in the continuous case in [14]. Since no results concerning profits are available in [14], we cannot make any comparaison in this issue.

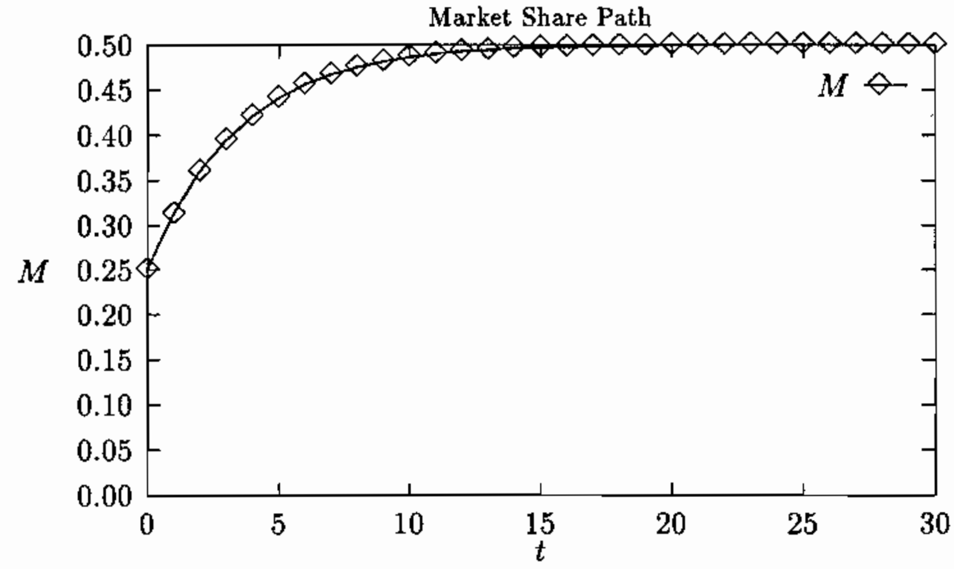

Figure 4: Market Share Path in the symetric case

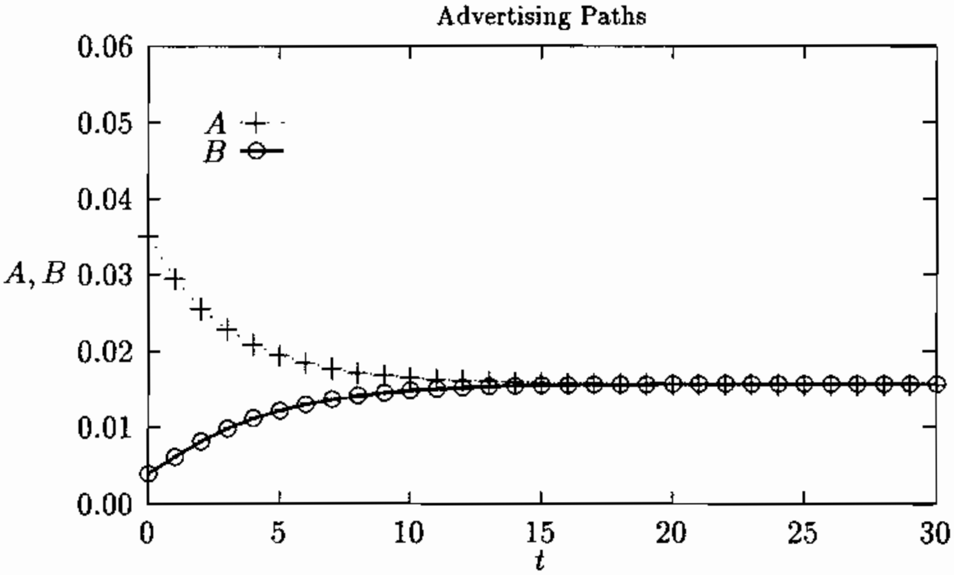

Figure 5: Advertising Paths in the symetric case

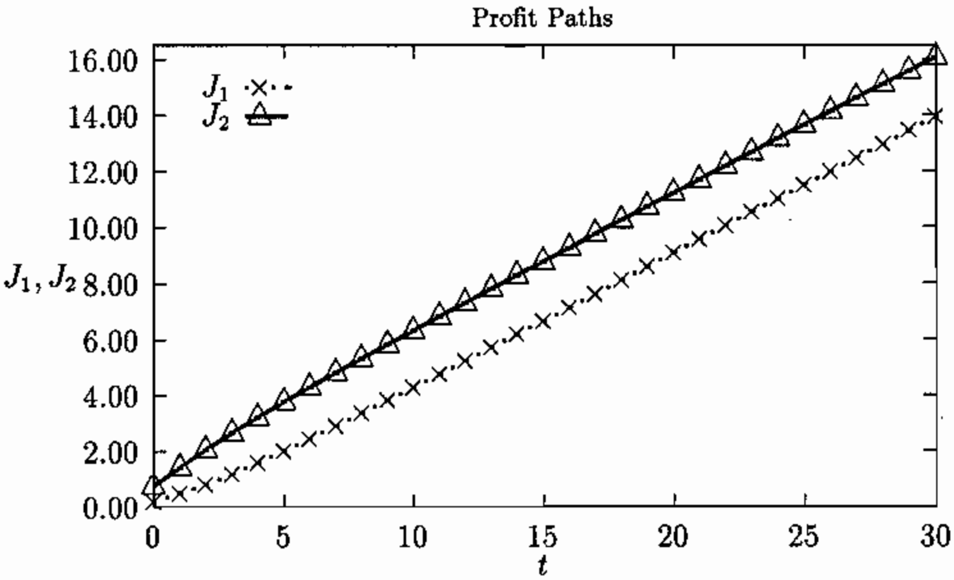

Figure 6: Profit Paths in the symetric case

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